- Reports & Notes

Reports & Notes

Beam Loading

Kai Meng Hock
Cockcroft Institute and Liverpool University

1 Introduction

This report is written with three objectives in mind:

1. to summarise what I have learnt about the basics of beam loading,

2. to explain how I have reproduced the calculation in [1] on 4GLS, and

3. to bridge the gap between undergrad physics and the beam loading literature.

As the beam loading model relies heavily on the concept of equivalent circuit, I shall start with a explanation on this in section 2. The calculation uses a number of parameters that requires familiarity with the rf cavity, and this is explained in section 3. In section 4, I explain how the beam current is related to the equivalent circuit model. In section 5, I show how the beam loading transient for bunch trains with gaps is calculated. Finally, in section 6, I explain what I have done to reproduce the calculations in [1].

2 Equivalent Circuit

Here I shall explain how rf generator, which I shall just call a klystron for convenience, together with the rf cavity, is modeled as an equivalent circuit. I assume here that there is no beam - this is discussed in sect. 4.

The klystron is connected to the rf cavity by a waveguide, which transmit electromagnetic wave to the cavity. It is important to know that the waveguide works by having electrical current flow over its inner wall. This electrical current generates the electromagnetic field. Without the current, the electromagnetic wave cannot just flow through the waveguide on its own. This is because of the boundary conditions that the fields must obey on the wall.

In the waveguide, the the electromagnetic wave is fully determined by the electric current. (Even the microwave that seems to propagate in free space is fully determined by the currents in the antenna generating it, in case you are wondering.) This also means that any effect of the electromagnetic wave, including acceleration of the beam, may be expressed in terms of the electric current. This is the physical basis of the equivalent circuit which the books never seem to mention.

For now, we ignore the electromagnetic wave completely and just think of the high frequency voltage generated by the klystron. This produces an alternating current which is carried by the waveguide to the rf cavity. A circuit as we know it has two wires coming out from the voltage source - one for the current to go out, one for the current to come back. A waveguide, however, can be just a hollow tube, so where is the circuit?

Figure 1 shows a schematic of a klystron. We can see that there is a pickup loop that goes out into a coaxial waveguide. This would eventually plug into the rf cavity. In between, there may be a hollow waveguide. The important thing is that the presence of the coaxial waveguide, even for just a short stretch, is sufficient to provide the picture of an equivalent circuit. We could think of the inner core and outer wall of the coaxial waveguide as the two wires coming out from a voltage source.

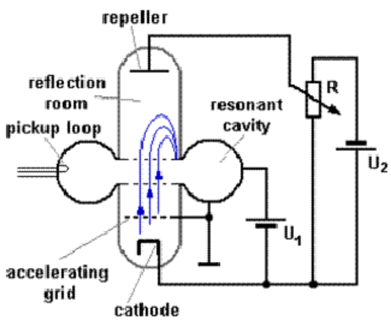


Figure 1: A reflex klystron [2]

We may now think of the klystron as a voltage source with, presumably, some resistance in series. This is picture is called the Thévenin equivalent, as shown in fig. 2. There is another picture which is, well, equivalent. This is the Norton equivalent. It consists of a current source with a resistance in parallel which produces the same effect as the Thévenin equivalent. The Norton equivalent is the one that is most commonly used in the literature for modelling beam loading.

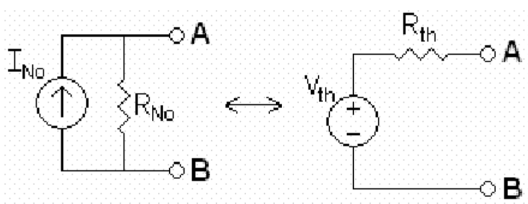


Figure 2: Norton (left) and Thévenin(right) equivalent circuit [3]

Next, we turn to the rf cavity. Again, the rf input to the cavity consist of a very similar structure to the pickup

loop in the klystron - see fig. 3. This section of the coaxial waveguide that plugs into the cavity again allows us to think about it as two wires entering and leaving the cavity.

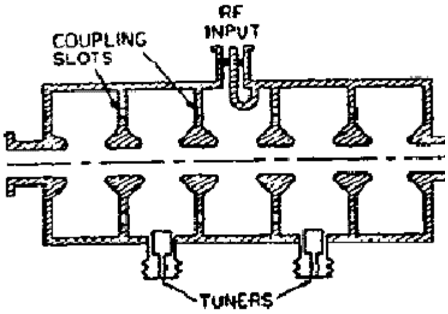


Figure 3: An rf cavity [4]

Here comes the tricky bit. What is the equivalent circuit for the cavity? There are two issues here:

First, the current from the klystron interacts with the current from the beam. We have so far pictured the current from the klystron as travelling along the outer wall and inner core of the transmission line waveguide joining it to the cavity. The beam going through the cavity, however, does not fit this picture. It travels through the vacuum. This issue would be treated in sect. 4.

Next, even if we ignore the beam, we have to model the cavity as some kind of lumped circuit elements that the current from the klystron would see. All books and papers that we can easily locate on beam loading would show us a resistance, a capacitance and an inductance in parallel. Reference [5] gives some details but I am not able to find a rigorous explanation anywhere. For now, I shall accept that this is a good but not exact model for the following reasons:

1. The RLC has a simple harmonic resonance. The rf cavity has many modes, but the fundamental mode probably looks sufficiently like a simple harmonic resonance. The model apparently works, since it is so widely used.

2. It is not exact because we know the rf cavity has Higher Order Modes with different frequencies, whereas the simple RLC circuit only has one frequency [6].

With this, I shall take the klystron-cavity part of the equivalent circuit on faith, since I am still looking for a publication that provides a thorough comparison of the equivalent circuit model with experiments. The equivalent circuit is shown in fig. 4.

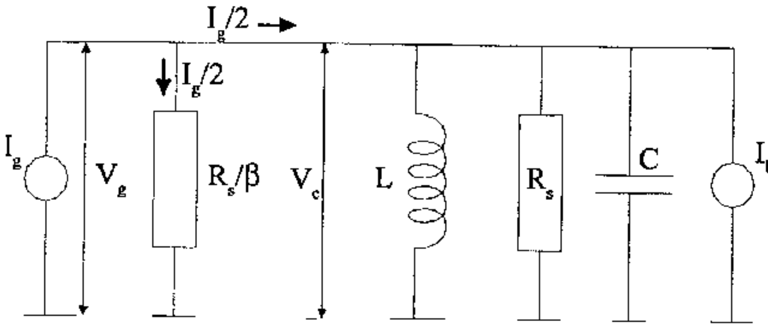


Figure 4: Equivalent circuit for rf generator and cavity [7]

I_g and V_g on the left form the klystron. L, R_s and C in the middle make up the rf cavity. I_b on the right represents the beam which is discussed in the next section. Note that R_s here is the shunt resistance of the cavity.

3 RF Cavity

Before going on to beam loading, it would be useful to become familiar with the basics of rf cavities. This would make it easier to understand and handle the parameters that are required for beam loading calculation. The ones that we would often see are listed in table 1. It should be sufficient for the present purpose to explain the basic physics of these terms. The full detail can be found in [7]. There are many others that need not be used in a calculation on beam loading for bunched charges. Their very existence makes it confusing and difficult to pick out the relevant parameters. When in doubt, references such as [4] and [7] should be consulted.

Table 1: Commonly used paramters for rf cavity

Parameter	Description
R_s	shunt impedance
φ	accelerating phase
V_{cy}	cavity voltage
Q	Q factor
ψ	tuning angle
ω_r	cavity resonance frequency

The meanings of these parameters are closely associated with the way that an rf cavity is used to accelerate a beam. A simple model that is commonly used for the rf cavity is the pill box cavity shown in fig. 5. The modes of electromagnetic field in this cavity happens to have analytic solutions because of the simple geometry. These modes may be excited, for instance, by making a small hole on the side of the cavity, and attaching the transmission line from the klystron as shown in fig. 3.

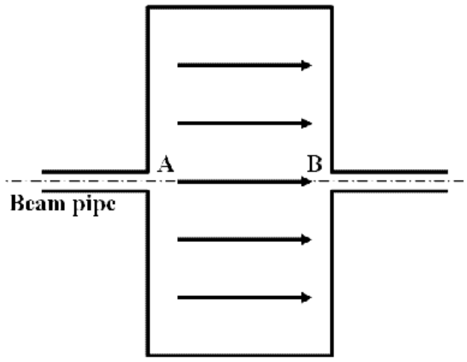


Figure 5: Pillbox model of the rf cavity

It so happens that the mode with the lowest frequency has a uniform electric field parallel to the axis. It is this field that can be used to accelerate the beam, by making a small hole at the centre of each flat face of the cavity, and joining a beam pipe to each hole, as shown in fig. 5. When a bunch enter the cavity, it would be in just the right direction to be accelerated by the electric field. The electric field oscillates sinusoidally, and half the time it would be in the wrong direction for the bunch. The bunch must be timed to enter at the right phase. This phase is called the accelerating phase, i.e. φ in table 1.

For this cavity, an analytic formula can be obtained for the power dissipated in the cavity [7]. This power is dissipated in the finite resistance on the cavity wall, as the current from the klystron flows around the wall. This analytic formula can be expressed in terms of the voltage across A and B in fig. 5. This volatge is important, because it is along AB that work is done on the bunch by the electric field. This voltage is called the cavity voltage. Let it be V_{cy}. It turns out that in the analytic formula [7], the power dissipated is directly related to V_{cy}^2, along with other parameters like cavity radius, rf frequency, etc. Let the power dissipated be P_{cy}. The formula, which is rather complex, may then be written in the form

$$P_{cy} = \frac{V_{cy}^2}{2R_s} \qquad (1)$$

with all the complexity of the formula absorbed into the R_s. Since this expression is the formula for power dissipated by a lumped resistance, we have effectively modelled the cavity as such. R_s may therefore be associated with the effective resistance of the cavity, provided the associated voltage V_{cy} is defined as the voltage across AB. If the whole thing looks very artificial, that is probably because it is. It is important to be aware of how the definitions are made up, so that we know what shunt impedance, which is the name given to R_s, really means.

We have explained half of table 1. The other half comes from modelling the cavity as a simple harmonic oscillator, which is done using the equivalent RLC circuit in sect. 2. In such a circuit, the dynamical variable (like the displacement of a pendulum bob) corresponds to the voltage from the klystron, V_g, that is applied to the cavity. The driving force corresponds to the klystron current, I_g. The relation can be written as a linear second order differential equation of the same form as a pendulum driven by an external force. For the case of the pendulum, we know that the external force need not be in phase with the displacement, unless it is at resonance. The same is true between I_g and V_g. The phase difference between them is called the tuning angle, ψ in table 1.

The resonance frequency ω_r is defined in the usual way. It is the frequency of the driving current I_g for which the klystron voltage V_g is at a maximum. At this point, one may start wondering if we need to know something about the klystron itself before we can talk about its current and vlotage. The good thing about the equivalent circuit is that we do not. Once we have modelled the klystron as a constant current source I_g, its voltage V_g is completely determined by what the klystron is connected to, in this case the rf cavity. In other words, the klystron is a battery, and we do not worry about what the chemicals in the battery are.

Finally, the Q factor is also defined in the usual way. There can be more than one ways. One example is 2π times the ratio of the stored energy to the energy loss per cycle of oscillation [8]. If we want to obtain a Q factor from the literature or from a colleague to use in a beam loading calculation, we need to distinguish between three commonly used Q factors: the unloaded Q factor Q_0, the loaded Q factor Q_L, and the external Q factor Q_{ext}. The unloaded Q factor is when we consider only the energy dissipated in the cavity. The loaded Q factor is when we include the energy dissipated in the klystron. A simple explanation for Q_{ext} can be found in [9]. Note that they are all different. In the literature, a parameter that is often given is R_s/Q. Here, the Q is the unloaded Q factor.

Cavities in general have more complex shapes. The above parameters must usually be computed numerically using softwares like SUPERFISH or HFSS.

4 Beam Loading

Here, we consider the issue of relating the current in the waveguide and cavity surfaces, which is represented by the wires in the equivalent circuit, to the beam current in the vacuum, which is not.

Since we are mainly interested in beam current carried by bunched charges, I shall consider the relation between a bunch of charged particles and the equivalent circuit. When a bunch passes through the vacuum in the cavity, it induces image charges on the cavity wall, together with the corresponding voltage. In order to relate the two, we need to know what this induced voltage is in terms of the bunch charge. This induced voltage is V_b, the voltage across I_b in fig. 4. If we can find a formula for V_b in terms of the bunch charge, this would provide the relation that we need. The derivation of this formula can be found in the section on fundamental theorem of beam loading in [4]. The formula is

$$V_b = 2kq \qquad (2)$$

where q is the bunch charge, and k is the loss parameter, given by

$$k = \frac{\omega}{4}(R_a/Q) \tag{3}$$

Here, ω is the cavity resonance frequency, R_a is the cavity shunt impedance, and Q is the Q factor. The meaning of these terms are explained in [4]. Except for the pillbox cavity, R_a/Q must usually be computed numerically using softwares like SUPERFISH or HFSS. Note that I have switched from the symbol R_s for shunt impedance in the previous section, to R_a here. This reflects the difference in notations in the references that I use. I shall keep switching notations, depending on which publication I happen to be citing. I make no attempt to standardise them. The difference in notations is a persistent source of confusion in the literature, and this is as good a place to introduce them as any.

Knowledge of the klystron voltage V_g and beam induced voltage V_b can be used to calculate the voltage at the cavity, as well as the voltage on the bunch. Knowledge of the cavity voltage can be used on feed forward system to compensate for its dip as a result of the bunch induced voltage, otherwise known as beam loading. Knowledge of the voltage on the bunch would tell us the energy gain. These calculations are carried out as follows:

The klystron voltage oscillates sinusoidally. Let V_g be the amplitude of this voltage. Suppose the bunch enters the cavity when the voltage is at a certain phase angle θ_g. Then the value of this voltage is $V_g\cos\theta_g$. Note that this voltage is the actual voltage along the path of the beam in the cavity - AB in fig. 5. In fact, this is how the equivalent circuit is to be defined, by choosing the appropriate I_g and R_s/β in fig. 4 so that V_g is equal to the voltage across AB. It should be mentioned here that there are two contributions fro this voltage, one from the klystron and one from the beam. I am referring to the former in this case. Likewise, the induced voltage as it is defined [4] is along this same path, and the in opposite direction. This constitutes the contribution from the beam. The resultant cavity voltage is then $V_g\cos\theta_g - 2kq$.

In beam loading literature, the above reasoning is usually expressed in terms of a phasor diagram, as shown in fig. 6. In this diagram, $V_g\cos\theta_g$ is represented as the horizontal component of a vector, [V\tilde]$_g$, or phasor as it is called here. θ_g is then the angle to the horizontal, described as "the phase of the generator voltage component just before the charge crosses the cavity reference plane"[4]. The induced voltage is represented by the horizontal phasor [V\tilde]$_{bo}$ pointing in the negative direction. The resultant phasor is [V\tilde]$_c^+$, and we are interested in its horizontal component, $V_g\cos\theta_g - 2kq$. In this simple case, a phasor diagram is not really necessary, but in more complex cases it could be useful [4].

While here, it is worth mentioning what the fundamental theorem of beam loading says - the charged particle sees half the voltage that it induces. So the net accelerting voltage on the charge is $V_g\cos\theta_g - kq$.

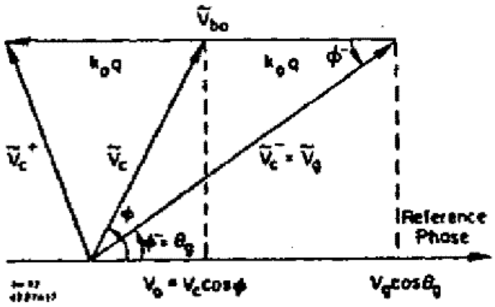


Figure 6: Single bunch phasor diagram [4]

5 Calculation

In the section, I shall explain how to understand and reproduce the beam loading calculation in [1]. Reference [1] calculates the beam loading transients, the dip in cavity voltage as a result of bunch trains going through the cavity. A bunch train is a number of equally spaced bunches. When this goes through the cavity, each bunch gives an induced voltage that lowers the cavity voltage slightly. The contribution from all bunches in the train are superposed to give the overall dip. After the train has passed, there would be a gap during which the cavity is empty. During this time, the cavity voltage would come back up. Before it can reach the maximum value V_g, however, the next train would usually have arrived to cause another dip in the cavity voltage. And so on for subsequent trains and gaps.

The objective here is to calculate how this cavity voltage changes. In order to do so, we need to know how the voltage induced by each bunch behaves after the bunch has left the cavity, and how the cavity voltage changes when it is empty after the train has left. Luckily, in both cases, the variation is exponential, with the same time constant known as the filling time, T_{fill}. There are some formulae for this parameter, but for now, I shall accept that it can be obtained by some software for EM computation. With this simple knowledge, we can go on the derive the formulae needed for the beam loading transient in [1].

First, we define the parameters that specifies the bunch train and gap in table 2.

Table 2: Bunch train and gap specifications

Parameter	Description
T_b	time between consecutive bunches in a train entering the cavity
n_t	number of bunches in the train
n_g	number of T_b in the gap
ω_{res}	cavity resonance frequency
R_{sh}	cavity shunt resistance
Q	cavity Q factor

Next, we define some derived parameters:

The fraction of bunch interval over fill time

$$\tau = T_b/T_{fill} \tag{4}$$

The cavity voltage is given by

$$V_c = V_g \cos\theta_g + V_b \tag{5}$$

where V_b is the beam induced voltage.

The voltage induced by a single bunch is

$$V_{b0} = -2kq \tag{6}$$

This induced voltage varies exponentially with a time constant T_{fill}. So at a time t after the bunch passes through the cavity (assuming that the time the bunch spends in the cavity is negligible compared to T_{fill}), the induced voltage becomes

$$V_b = V_{b0}\exp(-t/T_{fill}) \tag{7}$$

We can now start to derive the resultant V_b. Consider the bunch trains shown in fig. 7. Let n be the bunch number, and n_p be the total bunch spacings in a train and a gap, so that

$$n_p = n_t + n_g \tag{8}$$

First, we calculate the total induced voltage, V_{b0}^-, just before n=0 as follows.

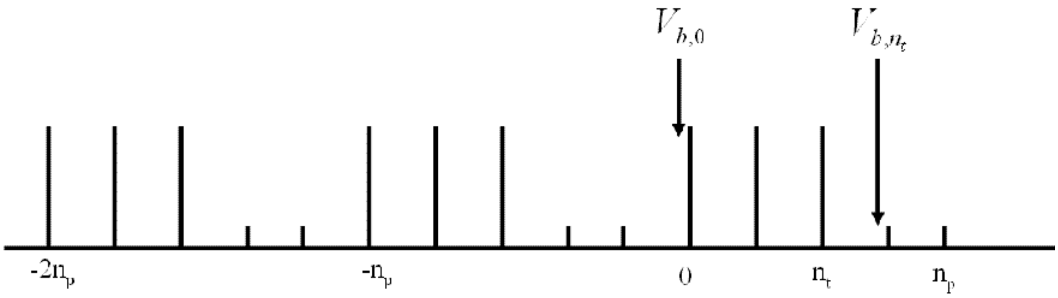


Figure 7: Bunch train. Long vertical line represents a bunch short vertical line a gap.

V_{b0}^- is the sum of the contributions from the exponentially decaying voltages induced by all the preceding bunches. Consider the contribution from the bunch at $n=-n_p$. The time from this to n=0 is $t=n_pT_b$, so the contribution is

$$V_{b0}\exp(-n_pT_b/T_{fill}) = V_{b0}\exp(-n_p\tau) \tag{9}$$

Likewise, the contribution from $n=-2n_p$ is $V_{b0}\exp(-2n_p\tau)$. The sum of all contributions from $n=-n_p, -2n_p, -3n_p$, ... is

$$\sum_{k=1}^{\infty} V_{b0}\exp(-kn_p\tau) = V_{b0}\sum_{k=1}^{\infty}\exp(-kn_p\tau) \tag{10}$$

Next, consider the contribution from $n=-n_p+1$. This would be $V_{b0}\exp(-(n_p-1)\tau)$. Now sum over all contributions from $n=-n_p+1, -2n_p+1, -3n_p+1$, ... This is

$$V_{b0}\sum_{k=1}^{\infty}\exp((-kn_p+1)\tau) = V_{b0}\,e^{\tau}\sum_{k=1}^{\infty}\exp(-kn_p\tau) \tag{11}$$

This is repeated for $n=-n_p+2$, $n=-n_p+3$, and so on until $n=-n_p+n_t-1$. For this last one, the sum over $n=-n_p+n_t-1, -2n_p+n_t-1, -3n_p+n_t-1$, ... gives

$$V_{b0}\sum_{k=1}^{\infty}\exp((-kn_p+n_t-1)\tau) = V_{b0}\exp((n_t-1)\tau)\sum_{k=1}^{\infty}\exp(-kn_p\tau) \tag{12}$$

The contributions from all preceding bunches, as shown in eqs. 10, 11 and 12, can now be summed to give

$$V_{b,0}^- = V_{b0}\sum_{m=0}^{n_t-1}\exp(m\tau)\sum_{k=1}^{\infty}\exp(-kn_p\tau) \tag{13}$$

$$= V_{b0} \frac{1-\exp(n_t\tau)}{1-\exp(\tau)} \frac{\exp(-n_p\tau)}{1-\exp(-n_p\tau)} \tag{14}$$

Next, we shall calculate the induced voltage just before n, when n is still within a bunch train, i.e. for $0 \le n \le n_t$. There are two sources of contribution to the induced voltage here.

One is from $V_{b,0}^-$, the induced voltage just before n=0, coming from all bunches before that. This is just going to continue decaying exponentially to give $V_{b,0}^- \exp(-n\tau)$.

The next contribution comes from all the preceding bunches within the same train, i.e. from 0 to n−1. Bunch 0 precedes bunch n by a time of nT_b, so it contributes $V_{b0} \exp(-(n-1)\tau)$. Likewise, bunch 1 gives $V_{b0} \exp(-(n-1)\tau)$ and so on, until bunch n−1, which gives $V_{b0} \exp(-\tau)$. Summing,

$$V_{b0} \exp(-n\tau) + V_{b0} \exp(-(n-1)\tau) + \ldots + V_{b0} \exp(-\tau) = V_{b0} \exp(-\tau) \frac{1-\exp(-n\tau)}{1-\exp(-\tau)} \tag{15}$$

The total induced voltage just before bunch n is then

$$V_{b,n}^- = V_{b,0}^- \exp(-n\tau) + V_{b0} \exp(-\tau) \frac{1-\exp(-n\tau)}{1-\exp(-\tau)} \tag{16}$$

for $0 \le n \le n_t$. Finally, we need to calculate the induced voltage when n is in the gap. This is for $n_t < n < n_t+n_g$.

First, let the induced voltage just before $n=n_t$ be V_{b,n_t}^-. The value of this can be obtained from eq. 16 by setting $n=n_t$.

Next, consider n somewhere within the gap. Since there is no bunch in the gap, the only contribution comes from V_{b,n_t}^- at position n_t, which decays exponentially to n over a time of $(n-n_t)T_b$.

The total induced voltage in the gap is then

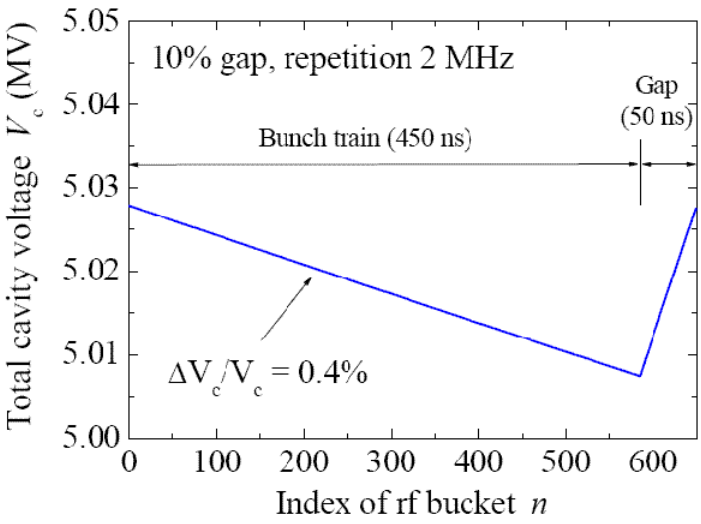
$$V_{b,n}^- = V_{b,n_t}^- \exp(-(n-n_t)\tau) \tag{17}$$

for $n_t < n < n_t+n_g$.

Equations 16 and 17 give the beam loading transient due to the bunch train and gap. They are listed in [1], and what I have done here is to supply the derivation. These two equations are used for to reproduce the calculation in [1] in the following section.

6 Some Discussions

In this section, I explain how I have reproduced one of the results in [1]. The result of my own calculation is shown in fig. 8, which agrees well with the corresponding result in [1]. It should have been a straight forward application of the formulae derived in sect. 5, except that there are some assumptions I need to make which should be mentioned here, together with the way I have used the parameters.



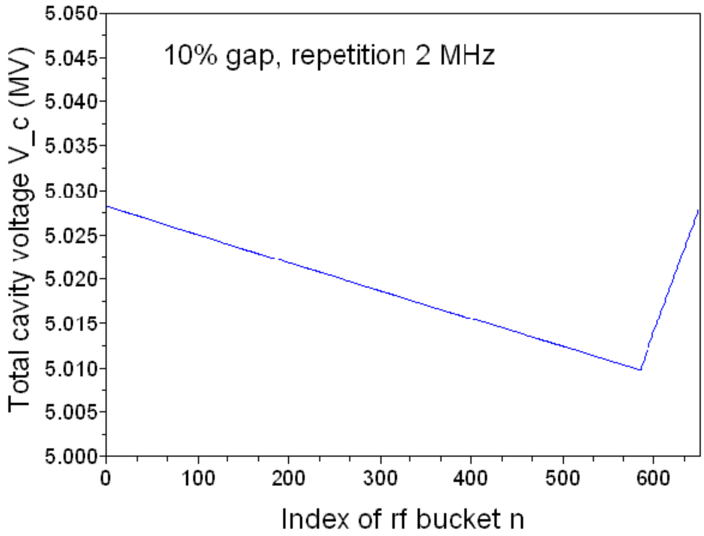


Figure 8: Transient voltage: Top panel from [1]. Bottom panel from me.

The cavity parameters used in [1] are given in table 3. These are the parameters for each of five identical cavities used to accelerate the bunches in [1].

Table 3: Injector cavity parameters [1]

Description	Parameter
RF voltage/cavity: V_C	1 MV
Accelerating phase: $\varphi = \cos^{-1}(V_a/V_c)$	0 radian
Beam power/cavity: $P_b=I_0V_c\cos\varphi$	100 kW
Cavity external-Q: $Q_{ext}=(Q_{ext})_{opt}= [(Q_0)/(\beta_{opt})]= [(V_c^2)/((R_{sh}/Q)P_b)]$	5×10^4
Cavity filling time: $T_f=2Q_L/\omega_{res}$	12 μs
Cavity tuning angle (=optimum): ψ	0

Since we can considereach bunch separately, we do not need to use the cavity tuning angle which relates the klystron current to the klystron voltage, V_g, which happens not to be given. Since the accelerating phase is 0, the klystron voltage when the bunch crosses the cavity is $V_g\cos\varphi = V_g$. The bunch induced voltage then gives a resultant $V_c=V_g+V_b$. I am thus skipping the definition for accelerating phase as stated in table 3, which is more properly defined only when V_b is sinusoidal.

V_b is given by the formulae in sect. 5. V_g is assumed to be a constant from the equivalent circuit model, once the frequency and circuit elements are specified. If this is the case, it means that $V_c=V_g+V_b$ is not a constant, since V_b varies with n. In that case, what does the V_c of 1 MV specified in table 3 mean?

I am going to assume that this means there is a feed forward system that adjusts the klystron voltage V_g automatically so that the resultant cavity voltage V_C is close to 1 MV. Since there are five cavities, this has to be multiplied by five to give 5 MV. V_b can be calculated using the formulae in sect. 5 and the parameters in table 3. We must remember to multiply this by five as well.

Finally, in order to reproduce the result in [1], shown on the top panel of fig. 8, I assume that the value set by the feed forward system is V_g=9.434 MV. This gives the result shown on the bottom panel of fig. 8, which agrees well with the top panel.

References

[1] Sakanaka, S., ERL07 Workshop Proceedings, Daresbury, 2007.
www.astec.ac.uk/ERL07/Presentations/Tuesday_Presentations/WG_2/Session_2
/WG2_Sakanaka_Ion_v1c.ppt
pfwww.kek.jp/ERLoffice/info/publications/erl07_iontrapping_sakanaka.pdf

[2] http://en.wikipedia.org/wiki/Klystron

[3] http://en.wikipedia.org/wiki/Equivalent_circuit
http://en.wikipedia.org/wiki/Th%C3%A9venin%27s_theorem

[4] Wilson, Perry B., SLAC-PUB-2884 (1991), pp. 7, 11, 13, 35-38.
http://www.slac.stanford.edu/cgi-wrap/getdoc/slac-pub-2884.pdf

[5] C. G. Montgomery, R. H. Dicke and E. M. Purcell, eds. Principles of Microwave Circuits. Radiation Laboratory Series Vol. 8 (McGraw-Hill, New York, 1948), Ch. 7.

[6] http://en.wikipedia.org/wiki/RLC_circuit

[7] H. Wiedemann, Particle Accelerator Physics II, 2nd ed., Springer-Verlag, Berlin (2003), Chapter 6

[8] http://en.wikipedia.org/wiki/Q_factor

[9] Q factor measurements, analog and digital, By Darko Kajfez
http://www.ee.olemiss.edu/darko/rfqmeas2b.pdf

File translated from TEX by TTH, version 3.81.
On 28 Jul 2008, 18:58.